

B-spline quasi-interpolant representations and sampling recovery of functions with mixed smoothness

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Abstract

Let $\xi = \{x^j\}_{j=1}^n$ be a grid of n points in the d -cube $\mathbb{I}^d := [0, 1]^d$, and $\Phi = \{\varphi_j\}_{j=1}^n$ a family of n functions on \mathbb{I}^d . We define the linear sampling algorithm $L_n(\Phi, \xi, \cdot)$ for an approximate recovery of a continuous function f on \mathbb{I}^d from the sampled values $f(x^1), \dots, f(x^n)$, by

$$L_n(\Phi, \xi, f) := \sum_{j=1}^n f(x^j) \varphi_j.$$

For the Besov class $B_{p,\theta}^\alpha$ of mixed smoothness α (defined as the unit ball of the Besov space $MB_{p,\theta}^\alpha$), to study optimality of $L_n(\Phi, \xi, \cdot)$ in $L_q(\mathbb{I}^d)$ we use the quantity

$$r_n(B_{p,\theta}^\alpha)_q := \inf_{H, \xi} \sup_{f \in B_{p,\theta}^\alpha} \|f - L_n(\Phi, \xi, f)\|_q,$$

where the infimum is taken over all grids $\xi = \{x^j\}_{j=1}^n$ and all families $\Phi = \{\varphi_j\}_{j=1}^n$ in $L_q(\mathbb{I}^d)$. We explicitly constructed linear sampling algorithms $L_n(\Phi, \xi, \cdot)$ on the grid $\xi = G^d(m) := \{(2^{-k_1} s_1, \dots, 2^{-k_d} s_d) \in \mathbb{I}^d : k_1 + \dots + k_d \leq m\}$, with Φ a family of linear combinations of mixed B-splines which are mixed tensor products of either integer or half integer translated dilations of the centered B-spline of order r . The grid $G^d(m)$ is of the size $2^m m^{d-1}$ and sparse in comparing with the generating dyadic coordinate cube grid of the size 2^{dm} . For various $0 < p, q, \theta \leq \infty$ and $1/p < \alpha < r$, we proved upper bounds for the worst case error $\sup_{f \in B_{p,\theta}^\alpha} \|f - L_n(\Phi, \xi, f)\|_q$ which coincide with the asymptotic order of $r_n(B_{p,\theta}^\alpha)_q$ in some cases. A key role in constructing these linear sampling algorithms, plays a quasi-interpolant representation of functions $f \in B_{p,\theta}^\alpha$ by mixed B-spline series with the coefficient functionals which are explicitly constructed as linear combinations of an absolute constant number of values of functions. Moreover, we proved that the quasi-norm of the Besov space $MB_{p,\theta}^\alpha$ is equivalent to a discrete quasi-norm in terms of the coefficient functionals.

Keywords Linear sampling algorithm · Quasi-interpolant · Quasi-interpolant representation · Mixed B-spline · Besov space of mixed smoothness.

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1 Introduction

The aim of the present paper is to investigate linear sampling algorithms for recovery of functions on the unit d -cube $\mathbb{I}^d := [0, 1]^d$, having a mixed smoothness. Let $\xi = \{x^j\}_{j=1}^n$ be a grid of n points in \mathbb{I}^d , and $\Phi = \{\varphi_j\}_{j=1}^n$ a family of n functions on \mathbb{I}^d . Then for a continuous function f on \mathbb{I}^d , we can define the linear sampling algorithm $L_n = L_n(\Phi, \xi, \cdot)$ for approximate recovering f from the sampled values $f(x^1), \dots, f(x^n)$, by

$$L_n(f) = L_n(\Phi, \xi, f) := \sum_{j=1}^n f(x^j) \varphi_j. \quad (1.1)$$

Let $L_q := L_q(\mathbb{I}^d)$, $0 < q \leq \infty$, denote the quasi-normed space of functions on \mathbb{I}^d with the q th integral quasi-norm $\|\cdot\|_q$ for $0 < q < \infty$, and the ess sup-norm $\|\cdot\|_\infty$ for $q = \infty$. The recovery error will be measured by $\|f - L_n(\Phi, \xi, f)\|_q$.

If W is a class of continuous functions, $\sup_{f \in B_{p,\theta}^\alpha} \|f - L_n(\Phi, \xi, f)\|_q$ is the worst case error of the recovery of functions f from W by the linear sampling algorithm $L_n(\Phi, \xi, \cdot)$. To study optimality of linear sampling algorithms of the form (1.1) for recovering $f \in W$ from n their values, we will use the quantity

$$r_n(W)_q := \inf_{\xi, \Phi} \sup_{f \in W} \|f - L_n(\Phi, \xi, f)\|_q, \quad (1.2)$$

where the infimum is taken over all grids $\xi = \{x^j\}_{j=1}^n$ and all families $\Phi = \{\varphi_j\}_{j=1}^n$ in L_q .

A challenging problem in linear sampling recovery of functions from a class W with a given mixed smoothness, is to construct a sampling algorithm $L_n(\Phi, \xi, \cdot)$ with an appropriate sampling grid $\xi = \{x^j\}_{j=1}^n$ and family $\Phi = \{\varphi_j\}_{j=1}^n$ which would be asymptotically optimal in terms of the quantity $r_n(W)_q$.

For periodic functions Smolyak [23] first constructed a specific linear sampling algorithm based on the de la Vallee Poussin kernel and the following dyadic grid in \mathbb{I}^d

$$G^d(m) := \{(2^{-k_1} s_1, \dots, 2^{-k_d} s_d) \in \mathbb{I}^d : k \in \Delta(m)\} = \{2^{-k} s : k \in \Delta(m), s \in I^d(k)\}.$$

Here and in what follows, we use the notations: $xy := (x_1 y_1, \dots, x_d y_d)$; $2^x := (2^{x_1}, \dots, 2^{x_d})$; $|x|_1 := \sum_{i=1}^d |x_i|$ for $x, y \in \mathbb{R}^d$; $\Delta(m) := \{k \in \mathbb{Z}_+^d : |k|_1 \leq m\}$; $I^d(k) := \{s \in \mathbb{Z}_+^d : 0 \leq s_i \leq 2^{k_i}, i \in N[d]\}$; $N[d]$ denotes the set of all natural numbers from 1 to d ; x_i denotes the i th coordinate of $x \in \mathbb{R}^d$, i.e., $x := (x_1, \dots, x_d)$. Temlyakov [24], [26], [27] and Dinh Dung [9]–[11] developed Smolyak's construction for study the asymptotic order of $r_n(W)_q$ for periodic Sobolev classes W_p^α and Hölder classes H_p^α as well their intersection. In particular, the first asymptotic order

$$r_n(H_p^\alpha)_q \asymp (n^{-1} \log^{d-1} n)^{\alpha-1/p+1/q} (\log^{d-1} n)^{1/q}, \quad 1 < p < q \leq 2, \quad \alpha > 1/p,$$

was obtained in [9]–[10]. For non-periodic functions of mixed smoothness $1/p < \alpha \leq 2$, this problem has been recently studied by Sickel and Ullrich [22], using the mixed tensor product of piecewise linear B-splines (of order 2) and the grid $G^d(m)$. It is interesting to notice that the linear sampling algorithms considered by above mentioned authors are interpolating at the grid $G^d(m)$.

Naturally, the quantity $r_n(W)_q$ of optimal linear sampling recovery is related to the problem of optimal linear approximation in terms of the linear n -width $\lambda_n(W)_q$ introduced by Tikhomirov [28]:

$$\lambda_n(W)_q := \inf_{A_n} \sup_{f \in W} \|f - A_n(f)\|_q,$$

where the infimum is taken over all linear operators A_n of rank n in L_q . The linear n -width $\lambda_n(W)_q$ was studied in [14], [20], [21], ect. for various classes W of functions with mixed smoothness. The inequality $r_n \geq \lambda_n$ is quite useful in investigation of the (asymptotic) optimality of a given linear sampling algorithm. It also allows to establish a lower bound of r_n via a known lower bound of λ_n .

In the present paper, we continue to research this problem. We will take functions to be recovered from the Besov class $B_{p,\theta}^\alpha$ of functions on \mathbb{I}^d , which is defined as the unit ball of the Besov space $MB_{p,\theta}^\alpha$ having mixed smoothness α . For functions in $B_{p,\theta}^\alpha$, we will construct linear sampling algorithms $L_n(\Phi, \xi, \cdot)$ on the grid $\xi = G^d(m)$ with Φ a family of linear combinations of mixed B-splines which are mixed tensor products of either integer or half integer translated dilations of the centered B-spline of order $r > \alpha$. We will be concerned with the worst case error of the recovery of $B_{p,\theta}^\alpha$ in the space L_q by these linear sampling algorithms and their asymptotic optimality in terms of the quantity $r_n(B_{p,\theta}^\alpha)_q$ for various $0 < p, q, \theta \leq \infty$ and $1/p \leq \alpha < r$. A key role in constructing these linear sampling algorithms, plays a quasi-interpolant representation of functions $f \in MB_{p,\theta}^\alpha$ by mixed B-spline series which will be explicitly constructed. Let us give a sketch of the main results of the present paper.

We first describe representations by mixed B-spline series constructed on the basic of quasi-interpolants. For a given natural number r , let M be the centered B-spline of order r with support $[-r/2, r/2]$ and knots at the points $-r/2, -r/2+1, \dots, r/2-1, r/2$. We define the integer translated dilation $M_{k,s}$ of M by

$$M_{k,s}(x) := M(2^k x - s), \quad k \in \mathbb{Z}_+, \quad s \in \mathbb{Z},$$

and the mixed d -variable B-spline $M_{k,s}$ by

$$M_{k,s}(x) := \prod_{i=1}^d M_{k_i, s_i}(x_i), \quad k \in \mathbb{Z}_+^d, \quad s \in \mathbb{Z}^d, \quad (1.3)$$

where \mathbb{Z}_+ is the set of all non-negative integers, $\mathbb{Z}_+^d := \{s \in \mathbb{Z}^d : s_i \geq 0, i \in N[d]\}$. Further, we define the half integer translated dilation $M_{k,s}^*$ of M by

$$M_{k,s}^*(x) := M(2^k x - s/2), \quad k \in \mathbb{Z}_+, \quad s \in \mathbb{Z},$$

and the mixed d -variable B-spline $M_{k,s}^*$ by

$$M_{k,s}^*(x) := \prod_{i=1}^d M_{k_i, s_i}^*(x_i), \quad k \in \mathbb{Z}_+^d, \quad s \in \mathbb{Z}^d.$$

In what follows, the B-spline M will be fixed. We will denote $M_{k,s}^r := M_{k,s}$ if the order r of M is even, and $M_{k,s}^r := M_{k,s}^*$ if the order r of M is odd.

Let $0 < p, \theta \leq \infty$, and $1/p < \alpha < \min(r, r - 1 + 1/p)$. Then we prove the following mixed B-spline quasi-interpolant representation of functions $f \in MB_{p,\theta}^\alpha$. Namely, a function f in the Besov space $MB_{p,\theta}^\alpha$ can be represented by the mixed B-spline series

$$f = \sum_{k \in \mathbb{Z}_+^d} \sum_{s \in J_r^d(k)} c_{k,s}^r(f) M_{k,s}^r, \quad (1.4)$$

converging in the quasi-norm of $MB_{p,\theta}^\alpha$, where $J_r^d(k)$ is the set of s for which $M_{k,s}^r$ do not vanish identically on \mathbb{I}^d , and the coefficient functionals $c_{k,s}^r(f)$ explicitly constructed as linear combinations of an absolute constant number of values of f which does not depend on neither k, s nor f . Moreover, we prove that the quasi-norm of $MB_{p,\theta}^\alpha$ is equivalent to some discrete quasi-norm in terms of the coefficient functionals $c_{k,s}^r(f)$. B-spline quasi-interpolant representations of functions from the isotropic Besov spaces has been constructed in [12], [13]. Different B-spline quasi-interpolant representations were considered in [7]. Both these representations were constructed on the basis of B-spline quasi-interpolants. The reader can see the books [2], [5] for survey and details on quasi-interpolants.

Let us construct linear sampling algorithms $L_n(\Phi, \xi, \cdot)$ on the grid $\xi = G^d(m)$ on the basis of the representation (1.4). For $m \in \mathbb{Z}_+$, let the linear operator R_m be defined for functions f on \mathbb{I}^d by

$$R_m(f) := \sum_{k \in \Delta(m)} \sum_{s \in J_r^d(k)} c_{k,s}^r(f) M_{k,s}^r. \quad (1.5)$$

If \bar{m} is the largest of m such that

$$2^m m^{d-1} \asymp |G^d(m)| \leq n$$

for a given n , where $|A|$ denotes the cardinality of A , then the operator $R_{\bar{m}}$ is a linear sampling algorithm of the form (1.1) on the grid $G^d(\bar{m})$. More precisely,

$$R_{\bar{m}}(f) = L_n(\Phi, \xi, f) = \sum_{(k,s) \in G^d(\bar{m})} f(2^{-k}s) \psi_{k,s},$$

where $\psi_{k,s}$ are explicitly constructed as linear combinations of an absolute constant of B-splines $M_{k,j}^r$, which does not depend on neither k, s nor f . It is worth to emphasize that the grid $G^d(m)$ is of the size $2^m m^{d-1}$ and sparse in comparing with the generating dyadic coordinate cube grid of the size 2^{dm} . We give now a brief of our results concerning with the worst case error of the recovery of functions f from $B_{p,\theta}^\alpha$ by the linear sampling algorithms $R_{\bar{m}}(f)$ and their asymptotic optimality.

We use the notations: $x_+ := \max(0, x)$ for $x \in \mathbb{R}$; $A_n(f) \ll B_n(f)$ if $A_n(f) \leq C B_n(f)$ with C an absolute constant not depending on n and/or $f \in W$, and $A_n(f) \asymp B_n(f)$ if $A_n(f) \ll B_n(f)$ and $B_n(f) \ll A_n(f)$. Let us introduce the abbreviations:

$$E(m) := \sup_{f \in B_{p,\theta}^\alpha} \|f - R_m(f)\|_q, \quad r_n := r_n(B_{p,\theta}^\alpha)_q.$$

Let $0 < p, q, \theta \leq \infty$ and $1/p < \alpha < r$. Then we have the following upper bound of r_n and $E(\bar{m})$.

(i) For $p \geq q$,

$$r_n \ll E(\bar{m}) \ll \begin{cases} (n^{-1} \log^{d-1} n)^\alpha, & \theta \leq \min(q, 1), \\ (n^{-1} \log^{d-1} n)^\alpha (\log^{d-1} n)^{1/q-1/\theta}, & \theta > \min(q, 1), \quad q \leq 1, \\ (n^{-1} \log^{d-1} n)^\alpha (\log^{d-1} n)^{1-1/\theta}, & \theta > \min(q, 1), \quad q > 1. \end{cases} \quad (1.6)$$

(ii) For $p < q$,

$$r_n \ll E(\bar{m}) \ll \begin{cases} (n^{-1} \log^{d-1} n)^{\alpha-1/p+1/q} (\log^{d-1} n)^{(1/q-1/\theta)_+}, & q < \infty, \\ (n^{-1} \log^{d-1} n)^{\alpha-1/p} (\log^{d-1} n)^{(1-1/\theta)_+}, & q = \infty. \end{cases} \quad (1.7)$$

From the embedding of $MB_{p,\theta}^\alpha$ into the isotropic Besov space of smoothness $d\alpha$ and known asymptotic order of the quantity (1.2) of its unit ball in L_q (see [8], [16], [17], [18], [27]) it follows that for $0 < p, q \leq \infty$, $0 < \theta \leq \infty$ and $\alpha > 1/p$, there always holds the lower bound $r_n \gg n^{-\alpha+(1/p-1/q)_+}$. However, this estimation is too rough and does not lead to the asymptotic order. By use of the inequality $\lambda_n(B_{p,\theta}^\alpha)_q \geq r_n$ and known results on $\lambda_n(B_{p,\theta}^\alpha)_q$ [14], [20], from (1.6) and (1.7) we obtain the asymptotic order of r_n for some cases. More precisely, we have the following asymptotic orders of r_n and $E(\bar{m})$ which show the asymptotic optimality of the linear sampling algorithms $R_{\bar{m}}$.

(i) For $p \geq q$ and $\theta \leq 1$,

$$E(\bar{m}) \asymp r_n \asymp (n^{-1} \log^{d-1} n)^\alpha, \quad \begin{cases} 2 \leq q < p < \infty, \\ 1 < p = q \leq \infty. \end{cases} \quad (1.8)$$

(ii) For $1 < p < q < \infty$,

$$E(\bar{m}) \asymp r_n \asymp (n^{-1} \log^{d-1} n)^{\alpha-1/p+1/q} (\log^{d-1} n)^{(1/q-1/\theta)_+}, \quad \begin{cases} 2 \leq p, \quad 2 \leq \theta \leq q, \\ q \leq 2. \end{cases} \quad (1.9)$$

The present paper is organized as follows. In Section 2, we give a necessary background of Besov spaces of mixed smoothness, B-spline quasi-interpolants, and prove a theorem on the mixed B-spline quasi-interpolant representation (1.4) and a relevant discrete equivalent quasi-norm for the Besov space of mixed smoothness $MB_{p,\theta}^\alpha$. In Section 3, we prove the upper bounds (1.6)–(1.7) and the asymptotic orders (1.8)–(1.9). In Section 4, we consider interpolant representations by series of the mixed tensor product of piecewise constant or piecewise linear B-splines. In Section 5, we present some auxiliary results.

2 B-spline quasi-interpolant representations

Let us introduce Besov spaces of functions with mixed smoothness and give necessary knowledge of them. For univariate functions the l th difference operator Δ_h^l is defined by

$$\Delta_h^l f(x) := \sum_{j=0}^l (-1)^{l-j} \binom{l}{j} f(x + jh).$$

If e is any subset of $N[d]$, for multivariate functions the mixed (l, e) th difference operator $\Delta_h^{l,e}$ is defined by

$$\Delta_h^{l,e} := \prod_{i \in e} \Delta_{h_i}^l, \quad \Delta_h^{l,\emptyset} = I,$$

where the univariate operator $\Delta_{h_i}^l$ is applied to the univariate function f by considering f as a function of variable x_i with the other variables held fixed. For a domain Ω in \mathbb{R}^d , denote by $L_p(\Omega)$ the quasi-normed space of functions on Ω with the p th integral quasi-norm $\|\cdot\|_{p,\Omega}$ for $0 < p < \infty$, and the ess sup-norm $\|\cdot\|_{\infty,\Omega}$ for $p = \infty$. Let

$$\omega_l^e(f, t)_p := \sup_{|h_i| < t_i, i \in e} \|\Delta_h^{l,e}(f)\|_{p, \mathbb{I}^d(h,e)}, \quad t \in \mathbb{I}^d,$$

be the mixed (l, e) th modulus of smoothness of f , where $\mathbb{I}^d(h, e) := \{x \in \mathbb{I}^d : x_i, x_i + lh_i \in \mathbb{I}, i \in e\}$ (in particular, $\omega_l^\emptyset(f, t)_p = \|f\|_p$). We will need the following modified (l, e) th mixed modulus of smoothness

$$w_l^e(f, t)_p := \left(\prod_{i \in e} t_i^{-1} \int_{U(t,e)} \int_{\mathbb{I}^d(h,e)} |\Delta_h^l(f, x)|^p dx dh \right)^{1/p},$$

where $U(t, e) := \{x \in \mathbb{I}^d : |x_i| \leq t, i \in e\}$. There hold the following inequalities

$$C_1 w_l^e(f, t)_p \leq \omega_l^e(f, t)_p \leq C_2 w_l^e(f, t)_p \quad (2.1)$$

with constants C_1, C_2 which depend on l, p, d only. A proof of these inequalities for the univariate modulus of smoothness is given in [19]. They can be proven in a similar way for the multivariate (l, e) th mixed modulus of smoothness.

If $0 < p, \theta \leq \infty$, $\alpha > 0$ and $l > \alpha$, we introduce the quasi-semi-norm $|f|_{B_{p,\theta}^{\alpha,e}}$ for functions $f \in L_p$ by

$$|f|_{MB_{p,\theta}^{\alpha,e}} := \begin{cases} \left(\int_{\mathbb{I}^d} \left\{ \prod_{i \in e} t_i^{-\alpha_i} \omega_l^e(f, t)_p \right\}^\theta \prod_{i \in e} t_i^{-1} dt \right)^{1/\theta}, & \theta < \infty, \\ \sup_{t \in \mathbb{I}^d} \prod_{i \in e} t_i^{-\alpha_i} \omega_l^e(f, t)_p, & \theta = \infty \end{cases}$$

(in particular, $|f|_{MB_{p,\theta}^{\alpha,\emptyset}} = \|f\|_p$).

For $0 < p, \theta \leq \infty$ and $0 < \alpha < l$, the Besov space $MB_{p,\theta}^\alpha$ is defined as the set of functions $f \in L_p$ for which the Besov quasi-norm $\|f\|_{MB_{p,\theta}^\alpha}$ is finite. The Besov quasi-norm is defined by

$$B(f) = \|f\|_{MB_{p,\theta}^\alpha} := \sum_{e \subset N[d]} |f|_{MB_{p,\theta}^{\alpha,e}}.$$

We will study the linear sampling recovery of functions from the Besov class

$$B_{p,\theta}^\alpha := \{f \in MB_{p,\theta}^\alpha : B(f) \leq 1\},$$

with the restriction on the smoothness $\alpha > 1/p$, which provides the compact embedding of $MB_{p,\theta}^\alpha$ into $C(\mathbb{I}^d)$, the space of continuous functions on \mathbb{I}^d with max-norm. We will also study this

problem for $B_{p,\theta}^\alpha$ with the restrictions $\alpha = 1/p$ and $p \leq \min(1, \theta)$ which is a sufficient condition for the continuous embedding of $MB_{p,\theta}^\alpha$ into $C(\mathbb{I}^d)$. In both these cases, $B_{p,\theta}^\alpha$ can be considered as a subset in $C(\mathbb{I}^d)$.

For any $e \subset N[d]$, put $\mathbb{Z}_+^d(e) := \{s \in \mathbb{Z}_+^d : s_i = 0, i \notin e\}$ (in particular, $\mathbb{Z}_+^d(\emptyset) = \{0\}$ and $\mathbb{Z}_+^d(N[d]) = \mathbb{Z}_+^d$). If $\{g_k\}_{k \in \mathbb{Z}_+^d(e)}$ is a sequence whose component functions g_k are in L_p , for $0 < p, \theta \leq \infty$ and $\beta \geq 0$ we define the $b_{\theta}^{\beta,e}(L_p)$ “quasi-norms”

$$\|\{g_k\}\|_{b_{\theta}^{\beta,e}(L_p)} := \left(\sum_{k \in \mathbb{Z}_+^d(e)} \left(2^{\beta|k|_1} \|g_k\|_p \right)^\theta \right)^{1/\theta}$$

with the usual change to a supremum when $\theta = \infty$. When $\{g_k\}_{k \in \mathbb{Z}_+^d(e)}$ is a positive sequence, we replace $\|g_k\|_p$ by $|g_k|$ and denote the corresponding quasi-norm by $\|\{g_k\}\|_{b_{\theta}^{\beta,e}}$.

For the Besov space $MB_{p,\theta}^\alpha$, from the definition and properties of the mixed (l, e) th modulus of smoothness it is easy to verify that there is the following quasi-norm equivalence

$$B(f) \asymp B_1(f) := \sum_{e \subset N[d]} \|\{\omega_l^e(f, 2^{-k})_p\}\|_{b_{\theta}^{\alpha,e}}.$$

Let $\Lambda = \{\lambda(s)\}_{s \in P(\mu)}$ be a finite even sequence, i.e., $\lambda(-j) = \lambda(j)$, where $P(\mu) := \{j \in \mathbb{Z} : |j| \leq \mu\}$ and $\mu \geq r/2 - 1$. We define the linear operator Q for functions f on \mathbb{R} by

$$Q(f, x) := \sum_{s \in \mathbb{Z}} \Lambda(f, s) M(x - s), \quad (2.2)$$

where

$$\Lambda(f, s) := \sum_{j \in P(\mu)} \lambda(j) f(s - j). \quad (2.3)$$

The operator Q is bounded in $C(\mathbb{R})$ and

$$\|Q(f)\|_{C(\mathbb{R})} \leq \|\Lambda\| \|f\|_{C(\mathbb{R})}$$

for each $f \in C(\mathbb{R})$, where

$$\|\Lambda\| = \sum_{j \in P(\mu)} |\lambda(j)|.$$

Moreover, Q is local in the following sense. There is a positive number $\delta > 0$ such that for any $f \in C(\mathbb{R})$ and $x \in \mathbb{R}$, $Q(f, x)$ depends only on the value $f(y)$ at an absolute constant number of points y with $|y - x| \leq \delta$. We will require Q to reproduce the space \mathcal{P}_{r-1} of polynomials of order at most $r - 1$, that is, $Q(p) = p$, $p \in \mathcal{P}_{r-1}$. An operator Q of the form (2.2)–(2.3) reproducing \mathcal{P}_{r-1} , is called a *quasi-interpolant in $C(\mathbb{R})$* .

There are many ways to construct quasi-interpolants. A method of construction via Neumann series was suggested by Chui and Diamond [3] (see also [2, p. 100–109]). A necessary and sufficient

condition of reproducing \mathcal{P}_{r-1} for operators Q of the form (2.2)–(2.3) with even r and $\mu \geq r/2$, was established in [1]. De Bore and Fix [4] introduced another quasi-interpolant based on the values of derivatives.

Let us give some examples of quasi-interpolants. The simplest example is a piecewise constant quasi-interpolant which is defined for $r = 1$ by

$$Q(f, x) := \sum_{s \in \mathbb{Z}} f(s) M(x - s),$$

where M is the symmetric piecewise constant B-spline with support $[-1/2, 1/2]$ and knots at the half integer points $-1/2, 1/2$. A piecewise linear quasi-interpolant is defined for $r = 2$ by

$$Q(f, x) := \sum_{s \in \mathbb{Z}} f(s) M(x - s), \quad (2.4)$$

where M is the symmetric piecewise linear B-spline with support $[-1, 1]$ and knots at the integer points $-1, 0, 1$. This quasi-interpolant is also called nodal and directly related to the classical Faber-Schauder basis. We will revisit it in Section 4. A quadric quasi-interpolant is defined for $r = 3$ by

$$Q(f, x) := \sum_{s \in \mathbb{Z}} \frac{1}{8} \{-f(s-1) + 10f(s) - f(s+1)\} M(x - s),$$

where M is the symmetric quadric B-spline with support $[-3/2, 3/2]$ and knots at the half integer points $-3/2, -1/2, 1/2, 3/2$. Another example is the cubic quasi-interpolant defined for $r = 4$ by

$$Q(f, x) := \sum_{s \in \mathbb{Z}} \frac{1}{6} \{-f(s-1) + 8f(s) - f(s+1)\} M(x - s),$$

where M is the symmetric cubic B-spline with support $[-2, 2]$ and knots at the integer points $-2, -1, 0, 1, 2$.

If Q is a quasi-interpolant of the form (2.2)–(2.3), for $h > 0$ and a function f on \mathbb{R} , we define the operator $Q(\cdot; h)$ by

$$Q(f; h) = \sigma_h \circ Q \circ \sigma_{1/h}(f),$$

where $\sigma_h(f, x) = f(x/h)$. By definition it is easy to see that

$$Q(f, x; h) = \sum_k \Lambda(f, k; h) M(h^{-1}x - k),$$

where

$$\Lambda(f, k; h) := \sum_{j \in P(\mu)} \lambda(j) f(h(k - j)).$$

The operator $Q(\cdot; h)$ has the same properties as Q : it is a local bounded linear operator in $C(\mathbb{R})$ and reproduces the polynomials from \mathcal{P}_{r-1} . Moreover, it gives a good approximation for smooth functions [5, p. 63–65]. We will also call it a *quasi-interpolant for $C(\mathbb{R})$* . However, the

quasi-interpolant $Q(\cdot; h)$ is not defined for a function f on \mathbb{I} , and therefore, not appropriate for an approximate sampling recovery of f from its sampled values at points in \mathbb{I} . An approach to construct a quasi-interpolant for functions on \mathbb{I} is to extend it by interpolation Lagrange polynomials. This approach has been proposed in [12] for the univariate case. Let us recall it.

For a non-negative integer k , we put $x_j = j2^{-k}, j \in \mathbb{Z}$. If f is a function on \mathbb{I} , let

$$U_k(f, x) := f(x_0) + \sum_{s=1}^{r-1} \frac{2^{sk} \Delta_{2^{-k}}^s f(x_0)}{s!} \prod_{j=0}^{s-1} (x - x_j),$$

$$V_k(f, x) := f(x_{2^k-r+1}) + \sum_{s=1}^{r-1} \frac{2^{sk} \Delta_{2^{-k}}^s f(x_{2^k-r+1})}{s!} \prod_{j=0}^{s-1} (x - x_{2^k-r+1+j})$$

be the $(r-1)$ th Lagrange polynomials interpolating f at the r left end points x_0, x_1, \dots, x_{r-1} , and r right end points $x_{2^k-r+1}, x_{2^k-r+3}, \dots, x_{2^k}$, of the interval \mathbb{I} , respectively. The function f_k is defined as an extension of f on \mathbb{R} by the formula

$$f_k(x) := \begin{cases} U_k(f, x), & x < 0, \\ f(x), & 0 \leq x \leq 1, \\ V_k(f, x), & x > 1. \end{cases}$$

Obviously, if f is continuous on \mathbb{I} , then f_k is a continuous function on \mathbb{R} . Let Q be a quasi-interpolant of the form (2.2)–(2.3) in $C(\mathbb{R})$. Put $\bar{\mathbb{Z}}_+ := \{k \in \mathbb{Z} : k \geq -1\}$. If $k \in \bar{\mathbb{Z}}_+$, we introduce the operator Q_k by

$$Q_k(f, x) = Q(f_k, x; 2^{-k}), \text{ and } Q_{-1}(f, x) := 0, \ x \in \mathbb{I},$$

for a function f on \mathbb{I} . We have for $k \in \mathbb{Z}_+$,

$$Q_k(f, x) = \sum_{s \in J(k)} a_{k,s}(f) M_{k,s}(x), \ \forall x \in \mathbb{I}, \quad (2.5)$$

where $J(k) := \{s \in \mathbb{Z} : -r/2 < s < 2^k + r/2\}$ is the set of s for which $M_{k,s}$ do not vanish identically on \mathbb{I} , and the coefficient functional $a_{k,s}$ is defined by

$$a_{k,s}(f) := \Lambda(f_k, s; 2^{-k}) = \sum_{|j| \leq \mu} \lambda(j) f_k(2^{-k}(s - j)).$$

Put $\bar{\mathbb{Z}}_+^d := \{k \in \mathbb{Z}_+^d : k_i \geq -1, \ i \in N[d]\}$. For $k \in \bar{\mathbb{Z}}_+^d$, let the mixed operator Q_k be defined by

$$Q_k := \prod_{i=1}^d Q_{k_i}, \quad (2.6)$$

where the univariate operator Q_{k_i} is applied to the univariate function f by considering f as a function of variable x_i with the other variables held fixed.

We have

$$Q_k(f, x) = \sum_{s \in J^d(k)} a_{k,s}(f) M_{k,s}(x), \quad \forall x \in \mathbb{I}^d,$$

where $M_{k,s}$ is the mixed B-spline defined in (1.3), $J^d(k) := \{s \in \mathbb{Z}^d : -r/2 < s_i < 2^{k_i} + r/2, i \in N[d]\}$ is the set of s for which $M_{k,s}$ do not vanish identically on \mathbb{I}^d ,

$$a_{k,s}(f) := a_{k_1, s_1}((a_{k_2, s_2}(\dots a_{k_d, s_d}(f))), \quad (2.7)$$

and the univariate coefficient functional a_{k_i, s_i} is applied to the univariate function f by considering f as a function of variable x_i with the other variables held fixed.

The operator Q_k is a local bounded linear mapping in $C(\mathbb{I}^d)$ and reproducing \mathcal{P}_{r-1}^d the space of polynomials of order at most $r-1$ in each variable x_i . More precisely, there is a positive number $\delta > 0$ such that for any $f \in C(\mathbb{I}^d)$ and $x \in \mathbb{I}^d$, $Q_k(f, x)$ depends only on the value $f(y)$ at an absolute constant number of points y with $|y_i - x_i| \leq \delta 2^{-k_i}, i \in N[d]$;

$$\|Q_k(f)\|_{C(\mathbb{I}^d)} \leq C \|\Lambda\|^d \|f\|_{C(\mathbb{I}^d)} \quad (2.8)$$

for each $f \in C(\mathbb{I}^d)$ with a constant C not depending on k ; and,

$$Q_k(p^*) = p, \quad p \in \mathcal{P}_{r-1}^d, \quad (2.9)$$

where p^* is the restriction of p on \mathbb{I}^d . The multivariate Q_k is called a *mixed quasi-interpolant* in $C(\mathbb{I}^d)$.

From (2.8) and (2.9) we can see that

$$\|f - Q_k(f)\|_{C(\mathbb{I}^d)} \rightarrow 0, \quad k \rightarrow \infty. \quad (2.10)$$

(Here and in what follows, $k \rightarrow \infty$ means that $k_i \rightarrow \infty$ for $i \in N[d]$).

If $k \in \bar{\mathbb{Z}}_+^d$, we define $T_k := I - Q_k$ for the univariate operator Q_k , where I is the identity operator. if $k \in \bar{\mathbb{Z}}_+^d$, we define the mixed operator T_k in the manner of the definition (2.6) by

$$T_k := \prod_{i=1}^d T_{k_i}.$$

For any $e \subset N[d]$, put $\bar{\mathbb{Z}}_+^d(e) := \{s \in \bar{\mathbb{Z}}_+^d : s_i > -1, i \in e, s_i = -1, i \notin e\}$ (in particular, $\bar{\mathbb{Z}}_+^d(\emptyset) = \{(-1, -1, \dots, -1)\}$ and $\bar{\mathbb{Z}}_+^d(N[d]) = \mathbb{Z}_+^d$). We have $\bar{\mathbb{Z}}_+^d(u) \cap \bar{\mathbb{Z}}_+^d(v) = \emptyset$ if $u \neq v$, and the following decomposition of $\bar{\mathbb{Z}}_+^d$:

$$\bar{\mathbb{Z}}_+^d = \bigcup_{e \subset N[d]} \bar{\mathbb{Z}}_+^d(e).$$

If τ is a number such that $0 < \tau \leq \min(p, 1)$, then for any sequence of functions $\{g_k\}$ there is the inequality

$$\left\| \sum g_k \right\|_p^\tau \leq \sum \|g_k\|_p^\tau. \quad (2.11)$$

Lemma 2.1 *Let $0 < p \leq \infty$ and $\tau \leq \min(p, 1)$. Then for any $f \in C(\mathbb{I}^d)$ and $k \in \bar{\mathbb{Z}}_+^d(e)$, there holds the inequality*

$$\|T_k(f)\|_p \leq C \left(\sum_{s \in \mathbb{Z}_+^d(e), s \geq k} \left\{ 2^{|s-k|_1/p} \omega_r^e(f, 2^{-s})_p \right\}^\tau \right)^{1/\tau} \quad (2.12)$$

with some constant C depending at most on r, μ, p, d and $\|\Lambda\|$, whenever the sum in the right-hand side is finite.

Proof. Notice that $\bar{\mathbb{Z}}_+^d(\emptyset) = \{(-1, -1, \dots, -1)\}$ and consequently, the inequality (2.12) is trivial for $e = \emptyset$: $\|f\|_p \leq C \omega_r^\emptyset(f, 1)_p = C \|f\|_p$. Consider the case where $e \neq \emptyset$. For simplicity we prove the lemma for $d = 2$ and $e = \{1, 2\}$, i.e., $\bar{\mathbb{Z}}_+^d(e) = \mathbb{Z}_+^2$. This lemma has proven in [12], [13] for univariate functions ($d = 1$) and even r . It can be proven for univariate functions and odd r in a completely similar way. Therefore, by (2.1) there holds the inequality

$$\|T_{k_i}(f)\|_p \ll \left(\sum_{s_i \geq k_i} \left\{ 2^{(s_i - k_i)/p} \left(2^{-s_i} \int_{U(2^{-s_i})} \int_{\mathbb{I}(h_i)} |\Delta_{h_i}^l(f, x)|^p dx_i dh_i \right)^{1/p} \right\}^\tau \right)^{1/\tau}, \quad i = 1, 2, \quad (2.13)$$

where the norm $\|T_{k_i}(f)\|_p$ is applied to the univariate function f by considering f as a function of variable x_i with the other variable held fixed.

If $1 \leq p < \infty$, we have by (2.13) applied for $i = 1$,

$$\begin{aligned} \|T_{k_1} T_{k_2}(f)\|_p &\ll \left(\int_{\mathbb{I}} \left\{ \sum_{s_1 \geq k_1} 2^{(s_1 - k_1)/p} \left(2^{-s_1} \int_{U(2^{-s_1})} \int_{\mathbb{I}(h_1)} |\Delta_{h_1}^l((T_{k_2}^1 f), x)|^p dx_1 dh_1 \right)^{1/p} \right\}^p dx_2 \right)^{1/p} \\ &\ll \sum_{s_1 \geq k_1} 2^{(s_1 - k_1)/p} \left(2^{-s_1} \int_{\mathbb{I}} \int_{U(2^{-s_1})} \int_{\mathbb{I}(h_1)} |\Delta_{h_1}^l((T_{k_2}^2 f), x)|^p dx_1 dh_1 dx_2 \right)^{1/p} \\ &= \sum_{s_1 \geq k_1} 2^{(s_1 - k_1)/p} \left(2^{-s_1} \int_{U(2^{-s_1})} \int_{\mathbb{I}(h_1)} \left\{ \int_{\mathbb{I}} |(T_{k_2}(\Delta_{h_1}^l f), x)|^p dx_2 \right\} dx_1 dh_1 \right)^{1/p}. \end{aligned}$$

Hence, applying (2.13) with $i = 2$ gives

$$\begin{aligned} \|T_{k_1} T_{k_2}(f)\|_p &\ll \sum_{s \geq k} 2^{|s-k|_1/p} \left(2^{-|s|_1} \int_{U(2^{-s})} \int_{\mathbb{I}^2(h)} |\Delta_h^l(f, x)|^p dx dh \right)^{1/p} \\ &\ll \sum_{s \geq k} 2^{|s-k|_1/p} \omega_r(f, 2^{-k})_p \ll \sum_{s \geq k} 2^{|s-k|_1/p} \omega_r(f, 2^{-k})_p. \end{aligned}$$

Thus, the lemma has proven when $1 \leq p < \infty$. The cases $0 < p < 1$ and $p = \infty$ can be proven in a similar way. \square

Let $J_r^d(k) := J^d(k)$ if r is even, and $J_r^d(k) := \{s \in \mathbb{Z}^d : -r < s_i < 2^{k_i+1} + r, i \in N[d]\}$ if r is odd. Notice that $J_r^d(k)$ is the set of s for which $M_{k,s}^r$ do not vanish identically on \mathbb{I}^d . Denote by $\Sigma_r^d(k)$ the span of the B-splines $M_{k,s}^r$, $s \in J_r^d(k)$. If $0 < p \leq \infty$, for all $k \in \mathbb{Z}_+^d$ and all $g \in \Sigma_r^d(k)$ such that

$$g = \sum_{s \in J_r^d(k)} a_s M_{k,s}^r, \quad (2.14)$$

there is the quasi-norm equivalence

$$\|g\|_p \asymp 2^{-|k|_1/p} \|\{a_s\}\|_{p,k}, \quad (2.15)$$

where

$$\|\{a_s\}\|_{p,k} := \left(\sum_{s \in J_r^d(k)} |a_s|^p \right)^{1/p}$$

with the corresponding change when $p = \infty$.

Let the mixed operator q_k , $k \in \mathbb{Z}_+^d$, be defined in the manner of the definition (2.6) by

$$q_k := \prod_{i=1}^d (Q_{k_i} - Q_{k_i-1}). \quad (2.16)$$

We have

$$Q_k = \sum_{k' \leq k} q_{k'}. \quad (2.17)$$

Here and in what follows, for $k, k' \in \mathbb{Z}^d$ the inequality $k' \leq k$ means $k'_i \leq k_i$, $i \in N[d]$. From (2.17) and (2.10) it is easy to see that a continuous function f has the decomposition

$$f = \sum_{k \in \mathbb{Z}_+^d} q_k(f) \quad (2.18)$$

with the convergence in the norm of $C(\mathbb{I}^d)$.

From the definition (2.16) and the refinement equation for the B-spline M , we can represent the component functions $q_k(f)$ as

$$q_k(f) = \sum_{s \in J_r^d(k)} c_{k,s}^r(f) M_{k,s}^r, \quad (2.19)$$

where $c_{k,s}^r$ are certain coefficient functionals of f , which are defined as follows. We first consider the univariate case. We have

$$q_k(f) = \sum_{s \in J(k)} a_{k,s}(f) M_{k,s} - \sum_{s \in J(k-1)} a_{k-1,s}(f) M_{k-1,s}. \quad (2.20)$$

If the order r of the B-spline M is even, by using the refinement equation

$$M(x) = 2^{-r+1} \sum_{j=0}^r \binom{r}{j} M(2x - j + r/2), \quad (2.21)$$

from (2.20) we obtain

$$q_k(f) = \sum_{s \in J_r(k)} c_{k,s}^r(f) M_{k,s}^r, \quad (2.22)$$

where

$$\begin{aligned} c_{k,s}^r(f) &:= a_{k,s}(f) - a'_{k,s}(f), \quad k > 0, \\ a'_{k,s}(f) &:= 2^{-r+1} \sum_{(m,j) \in C(k,s)} \binom{r}{j} a_{k-1,m}(f), \quad k > 0, \quad a'_{0,s}(f) := 0. \end{aligned} \quad (2.23)$$

and

$$C_r(k, s) := \{(m, j) : 2m + j - r/2 = s, \quad m \in J(k-1), \quad 0 \leq j \leq r\}, \quad k > 0, \quad C_r(0, s) := \{0\}.$$

If the order r of the B-spline M is odd, by using (2.21) from (2.20) we get (2.22) with

$$c_{k,s}^r(f) := \begin{cases} 0, & k = 0, \\ a_{k,s/2}(f), & k > 0, \quad s \text{ even}, \\ 2^{-r+1} \sum_{(m,j) \in C_r(k,s)} \binom{r}{j} a_{k-1,m}(f), & k > 0, \quad s \text{ odd}, \end{cases}$$

where

$$C_r(k, s) := \{(m, j) : 4m + 2j - r = s, \quad m \in J(k-1), \quad 0 \leq j \leq r\}, \quad k > 0, \quad C_r(0, s) := \{0\}.$$

In the multivariate case, the representation (2.19) holds true with the $c_{k,s}^r$ which are defined in the manner of the definition (2.7) by

$$c_{k,s}^r(f) = c_{k_1,s_1}^r((c_{k_2,s_2}^r(\dots c_{k_d,s_d}^r(f))). \quad (2.24)$$

Let us use the notations: $\mathbf{1} := (1, \dots, 1) \in \mathbb{R}^d$; $x_+ := ((x_1)_+, \dots, (x_d)_+)$ for $x \in \mathbb{R}^d$; $\mathbb{N}^d(e) := \{s \in \mathbb{Z}_+^d : s_i > 0, \quad i \in e, \quad s_i = 0, \quad i \notin e\}$ for $e \subset N[d]$ (in particular, $\mathbb{N}^d(\emptyset) = \{0\}$ and $\mathbb{N}^d(N[d]) = \mathbb{N}^d$). We have $\mathbb{N}^d(u) \cap \mathbb{N}^d(v) = \emptyset$ if $u \neq v$, and the following decomposition of \mathbb{Z}_+^d :

$$\mathbb{Z}_+^d = \bigcup_{e \subset N[d]} \mathbb{N}^d(e).$$

Lemma 2.2 *Let $0 < p \leq \infty$ and $\tau \leq \min(p, 1)$. Then for any $f \in C(\mathbb{I}^d)$ and $k \in \mathbb{N}^d(e)$, there holds the inequality*

$$\|q_k(f)\|_p \leq C \sum_{v \supset e} \left(\sum_{s \in \mathbb{Z}_+^d(v), \quad s \geq k} \left\{ 2^{|s-k|_1/p} \omega_r^v(f, 2^{-s})_p \right\}^\tau \right)^{1/\tau}$$

with some constant C depending at most on r, μ, p, d and $\|\Lambda\|$, whenever the sum in the right-hand side is finite.

Proof. From the equality

$$q_k = \prod_{i=1}^d (T_{k_i-1}^i - T_{k_i}^i),$$

it follows that

$$q_k = \sum_{u \subset N[d]} (-1)^{|u|} \prod_{i \in u} T_{k_i}^i \prod_{i \notin u} T_{k_i-1}^i = \sum_{u \subset N[d]} (-1)^{|u|} T_{k^u},$$

where k^u is defined by $k_i^u = k_i$ if $i \in u$, and $k_i^u = k_i - 1$ if $i \notin u$. Hence,

$$\|q_k(f)\|_p \ll \sum_{u \subset N[d]} \|T_{k^u}(f)\|_p. \quad (2.25)$$

Notice that $k^u \in \bar{\mathbb{Z}}_+^d(v)$ for some $v \supset e$, and $0 \leq k - k_+^u \leq k - k^u \leq \mathbf{1}$. Moreover, for $s \in \mathbb{Z}_+^d(v)$, $s \geq k^u$ if only if $s \geq k_+^u$. Hence, by Lemma 2.1 and properties of mixed modulus smoothness we have

$$\begin{aligned} \|T_{k^u}(f)\|_p &\ll \left(\sum_{s \in \mathbb{Z}_+^d(v), s \geq k^u} \left\{ 2^{|s-k^u|_1/p} \omega_r^v(f, 2^{-s})_p \right\}^\tau \right)^{1/\tau} \\ &\ll \left(\sum_{s \in \mathbb{Z}_+^d(v), s \geq k_+^u} \left\{ 2^{|s-k_+^u|_1/p} \omega_r^v(f, 2^{-s})_p \right\}^\tau \right)^{1/\tau} \\ &= \left(\sum_{s' \in \mathbb{Z}_+^d(v), s' \geq k} \left\{ 2^{|s'-k|_1/p} \omega_r^v(f, 2^{-(s'-(k-k_+^u))})_p \right\}^\tau \right)^{1/\tau} \\ &\ll \left(\sum_{s \in \mathbb{Z}_+^d(v), s \geq k} \left\{ 2^{|s-k|_1/p} \omega_r^v(f, 2^{-s})_p \right\}^\tau \right)^{1/\tau}. \end{aligned}$$

The last inequality together with (2.25) proves the lemma. \square

Lemma 2.3 *Let $0 < p \leq \infty$, $0 < \tau \leq \min(p, 1)$, $\delta = \min(r, r - 1 + 1/p)$. Then for any $f \in C(\mathbb{I}^d)$ and $k \in \mathbb{Z}_+^d(e)$, there holds the inequality*

$$\omega_r^e(f, 2^{-k})_p \leq C \left(\sum_{s \in \mathbb{Z}_+^d} \left\{ 2^{-\delta|(k-s)_+|_1} \|q_s(f)\|_p \right\}^\tau \right)^{1/\tau}$$

with some constant C depending at most on r, μ, p, d and $\|\Lambda\|$, whenever the sum in the right-hand side is finite.

Proof. For simplicity we prove the lemma for $e = N[d]$, i.e., $\mathbb{Z}_+^d(e) = \mathbb{Z}_+^d$. Let $f \in C(\mathbb{I}^d)$ and $k \in \mathbb{Z}_+^d$. From (2.18) and (2.11) we obtain

$$\|\Delta_h^r(f)\|_p \leq C \left(\sum_{s \in \mathbb{Z}_+^d} \|\Delta_h^r(q_s(f))\|_p^\tau \right)^{1/\tau}. \quad (2.26)$$

Further, by (2.19) we get

$$\Delta_h^r(q_k(f)) = \sum_{j \in J^d(s)} c_{s,j}^r(f) \Delta_h^r(M_{s,j}^r).$$

Notice that for any x , the number of non-zero B-spines in (2.19) is an absolute constant depending on r, d only. Thus, we have

$$|\Delta_h^r(q_s(f), x)|^p \ll \sum_{j \in J^d(s)} |c_{s,j}^r(f)|^p |\Delta_h^r(M_{s,j}^r, x)|^p, \quad x \in \mathbb{I}^d. \quad (2.27)$$

From properties of the B-spline M it is easy to prove the following estimate

$$\int_{\mathbb{I}^d(h)} |\Delta_h^r(M_{s,j}^r, x)|^p dx \ll 2^{-|s|_1 - \delta p(-\log|h|-s)_+ + |1|},$$

where we used the abbreviation $\log|h| := (\log|h_1|, \dots, \log|h_d|)$. Hence, by (2.27) we obtain

$$\begin{aligned} \|\Delta_h^r(q_s(f))\|_p &\ll 2^{-\delta|(-\log|h|-s)_+ + |1|} 2^{-|s|_1} \left(\sum_{j \in J^d(s)} |c_{s,j}^r(f)|^p \right)^{1/p} \\ &\ll 2^{-\delta|(-\log|h|-s)_+ + |1|} \|q_s(f)\|_p. \end{aligned}$$

By (2.26) we have

$$\|\Delta_h^r(f)\|_p \ll \left(\sum_{s \in \mathbb{Z}_+^d} \left\{ 2^{-\delta|(-\log|h|-s)_+ + |1|} \|q_s(f)\|_p \right\}^\tau \right)^{1/\tau}.$$

From the last inequality we prove the lemma. \square

For functions f on \mathbb{I}^d , we introduce the following quasi-norms:

$$\begin{aligned} B_2(f) &:= \|\{q_k(f)\}\|_{b_\theta^\alpha(L_p)}; \\ B_3(f) &:= \left(\sum_{k=0}^{\infty} (2^{(\alpha-d/p)k} \|\{c_{k,s}^r(f)\}\|_{p,k})^\theta \right)^{1/\theta}. \end{aligned}$$

We will need the following discrete Hardy inequality. Let $\{a_k\}_{k \in \mathbb{Z}_+^d}$ and $\{b_k\}_{k \in \mathbb{Z}_+^d}$ be two positive sequences and let for some $M > 0$, $\tau > 0$

$$b_k \leq M \left(\sum_{s \in \mathbb{Z}_+^d} \left(2^{\delta|(k-s)_+ + |1|} a_s \right)^\tau \right)^{1/\tau}. \quad (2.28)$$

Then for any $0 < \beta < \delta$, $\theta > 0$,

$$\|\{b_k\}\|_{b_\theta^\beta} \leq CM \|\{a_k\}\|_{b_\theta^\beta} \quad (2.29)$$

with $C = C(\beta, \theta, d)$. For a proof of this inequality for the univariate case see, e.g, [6]. In the general case it can be proven by induction based on the univariate case.

Theorem 2.1 *Let $0 < p, \theta \leq \infty$ and $1/p < \alpha < r$. Then the hold the following assertions.*

(i) *A function $f \in MB_{p,\theta}^\alpha$ can be represented by the mixed B-spline series*

$$f = \sum_{k \in \mathbb{Z}_+^d} q_k(f) = \sum_{k \in \mathbb{Z}_+^d} \sum_{s \in J_r^d(k)} c_{k,s}^r(f) M_{k,s}^r, \quad (2.30)$$

satisfying the convergence condition

$$B_2(f) \asymp B_3(f) \ll B(f),$$

where the coefficient functionals $c_{k,s}^r(f)$ are explicitly constructed by formula (2.23)–(2.24) as linear combinations of an absolute constant number of values of f which does not depend on neither k, s nor f .

(ii) *If in addition, $\alpha < \min(r, r - 1 + 1/p)$, then a continuous function f on \mathbb{I}^d belongs to the Besov space $MB_{p,\theta}^\alpha$ if and only if f can be represented by the series (2.30). Moreover, the Besov quasi-norm $B(f)$ is equivalent to one of the quasi-norms $B_2(f)$ and $B_3(f)$.*

Proof. Since by (2.15) the quasi-norms $B_2(f)$ and $B_3(f)$ are equivalent, it is enough to prove (i) and (ii) for $B_3(f)$. Fix a number $0 < \tau \leq \min(p, 1)$.

Assertion (i): For $k \in \mathbb{Z}_+^d$, put

$$b_k := 2^{|k|_1/p} \|q_k(f)\|_p, \quad a_k := \left(\sum_{v \supseteq e} \left\{ 2^{|k|_1/p} \omega_r^v(f, 2^{-k})_p \right\}^\tau \right)^{1/\tau}$$

if $k \in \mathbb{N}^d(e)$. By Lemma 2.2 we have for $k \in \mathbb{Z}_+^d$,

$$b_k \leq C \left(\sum_{s \geq k}^\infty a_s^\tau \right)^{1/\tau}.$$

Then applying the mixed discrete Hardy inequality (2.28)–(2.29) with $\beta = \alpha - 1/p$, gives

$$B_3(f) = \|\{b_k\}\|_{b_\theta^\beta} \leq C \|\{a_k\}\|_{b_\theta^\beta} \asymp B_1(f) \asymp B(f).$$

Assertion (ii): Let in addition, $\alpha < \min(r, r - 1 + 1/p)$. For $k \in \mathbb{Z}_+^d$, put

$$b_k := \left(\sum_{v \supseteq e} \left\{ \omega_r^v(f, 2^{-k})_p \right\}^\tau \right)^{1/\tau} \quad a_k := \|q_k(f)\|_p$$

if $k \in \mathbb{N}^d(e)$. By Lemma 2.3 we have for $k \in \mathbb{Z}_+^d$

$$b_k \leq C \left(\sum_{s \in \mathbb{Z}_+^d} \left(2^{\delta|(k-s)+|1} a_s \right)^\tau \right)^{1/\tau},$$

where $\delta = \min(r, r-1+1/p)$. Then applying the mixed discrete Hardy inequality (2.28)–(2.29) with $\beta = \alpha$, gives

$$B(f) \asymp B_1(f) \asymp \|\{b_k\}\|_{b_\theta^\beta} \leq C \|\{a_k\}\|_{b_\theta^\beta} = B_3(f).$$

The assertion (ii) is proven. \square

Remark From (2.23)–(2.24) we can see that if r is even, for each pair k, s the coefficient $c_{k,s}^r(f)$ is a linear combination of the values $f(2^{-k}(s-j))$, and $f(2^{-k+1}(s'-j))$, $j \in P^d(\mu)$, $s' \in C_r(k, s)$. The number of these values does not exceed the fixed number $(2\mu+1)^d((r+1)^d+1)$. If r is odd, we can say similarly about the coefficient $c_{k,s}^r(f)$.

3 Sampling recovery

Recall that the linear operator $R_m, m \in \mathbb{Z}_+$, is defined for functions on \mathbb{I}^d in (1.5) as follows.

$$R_m(f) = \sum_{k \in \Delta(m)} q_k(f) = \sum_{k \in \Delta(m)} \sum_{s \in J_r^d(k)} c_{k,s}^r(f) M_{k,s}^r. \quad (3.1)$$

Lemma 3.1 *For functions f on \mathbb{I}^d , R_m defines a linear sampling algorithm of the form (1.1) on the grid $G^d(m)$. More precisely,*

$$R_m(f) = L_n(f) = \sum_{(k,s) \in G^d(m)} f(2^{-k}j) \psi_{k,j},$$

where

$$n := |G^d(m)| = \sum_{k \in \Delta(m)} |I^d(k)| \asymp 2^m m^{d-1}; \quad (3.2)$$

$\psi_{k,j}$ are explicitly constructed as linear combinations of at most $(4\mu+r+5)^d$ B -splines $M_{k,s}^r \in M_r^d(m)$ for even r , and $(12\mu+2r+13)^d$ B -splines $M_{k,s}^r \in M_r^d(m)$ for odd r ; $M_r^d(m) := \{M_{k',s'}^r : k' \in \Delta(m), s' \in J_r^d(k')\}$.

Proof. Let us prove the lemma for even r . For odd r it can be proven in a similar way. For univariate functions the coefficient functionals $a_{k,s}(f)$ can be rewritten as

$$a_{k,s}(f) = \sum_{|s-j| \leq \mu} \lambda(s-j) f_k(2^{-k}j) = \sum_{j \in P(k,s)} \lambda_{k,s}(j) f(2^{-k}j),$$

where $\lambda_{k,s}(j) := \lambda(s-j)$ and $P(k, s) = P_s(\mu) := \{j \in \{0, 2^k\} : s-j \in P(\mu)\}$ for $\mu \leq s \leq 2^k - \mu$; $\lambda_{k,s}(j)$ is a linear combination of no more than $\max(r, 2\mu+1) \leq 2\mu+2$ coefficients $\lambda(j)$, $j \in P(\mu)$, for $s < \mu$ or $s > 2^k - \mu$ and

$$P(k, s) \subset \begin{cases} P_s(\mu) \cup \{0, r-1\}, & s < \mu, \\ P_s(\mu) \cup \{2^k - r + 1, 2^k\}, & s > 2^k - \mu. \end{cases}$$

Further, for univariate functions we have

$$\begin{aligned} c_{k,s}^r(f) &= \sum_{j \in P(k,s)} \lambda_{k,s}(j) f(2^{-k}j) - 2^{-r+1} \sum_{(m,\nu) \in C_r(k,s)} \binom{r}{\nu} \sum_{j \in P(k-1,m)} \lambda_{k-1,m}(j) f(2^{-k}(2j)) \\ &= \sum_{j \in G(k,s)} \lambda_{k,s}(j) f(2^{-k}j), \end{aligned}$$

where $G(k, s) := P(k, s) \cup \{2j : j \in P(k-1, m), (m, \nu) \in C(k, s)\}$. If $j \in P(k, s)$, we have $|j-s| \leq \max(r, 2\mu+1) \leq 2\mu+2$. If $j \in P(k-1, m)$, $(m, \nu) \in C(k, s)$, we have $|2j-s| = |2j-2m-\nu+r/2| \leq 2|j-m| + |\nu-r/2| \leq 2\max(r, 2\mu+1) + r+1 \leq 4\mu+r+5 =: \bar{\mu}$. Therefore, $G(k, s) \subset P_s(\bar{\mu})$, and we can rewrite the coefficient functionals $c_{k,s}^r(f)$ in the form

$$c_{k,s}^r(f) = \sum_{j-s \in P(\bar{\mu})} \lambda_{k,s}(j) f(2^{-k}j)$$

with zero coefficients $\lambda_{k,s}(j)$ for $j \notin G(k, s)$. Therefore, we have

$$\begin{aligned} q_k(f) &= \sum_{s \in J_r(k)} c_{k,s}^r(f) M_{k,s}^r = \sum_{s \in J_r(k)} \sum_{j-s \in P(\bar{\mu})} \lambda_{k,s}(j) f(2^{-k}j) M_{k,s}^r \\ &= \sum_{j \in I(k)} f(2^{-k}j) \sum_{s-j \in P(\bar{\mu})} \gamma_{k,j}(s) M_{k,s}^r \end{aligned}$$

for certain coefficients $\gamma_{k,j}(s)$. Thus, the univariate $q_k(f)$ is of the form

$$q_k(f) = \sum_{j \in I(k)} f(2^{-k}j) \psi_{k,j},$$

where

$$\psi_{k,j} := \sum_{s-j \in P(\bar{\mu})} \gamma_{k,j}(s) M_{k,s}^r,$$

are a linear combination of no more than the absolute number $4\mu+r+5$ of B-splines $M_{k,s}^r$, and the size $|I(k)|$ is 2^k . Hence, the multivariate $q_k(f)$ is of the form

$$q_k(f) = \sum_{j \in I^d(k)} f(2^{-k}j) \psi_{k,j},$$

where

$$\psi_{k,j} := \prod_{i=1}^d \psi_{k_i, j_i}$$

are a linear combination of no more than the absolute number $(4\mu + r + 5)^d$ of B-splines $M_{k,s}^r \in M_r^d(m)$, and the size $|I^d(k)|$ is $2^{|k|_1}$. From (3.1) we can see that $R_m(f)$ is of the form (1.1) with n as in (3.2). \square

Theorem 3.1 *Let $0 < p, q, \theta \leq \infty$ and $1/p < \alpha < r$. Then we have the following upper bound of $E(m)$.*

(i) For $p \geq q$,

$$E(m) \ll \begin{cases} 2^{-\alpha m}, & \theta \leq \min(q, 1), \\ 2^{-\alpha m} m^{(d-1)(1/q-1/\theta)}, & \theta > \min(q, 1), \quad q \leq 1, \\ 2^{-\alpha m} m^{(d-1)(1-1/\theta)}, & \theta > \min(q, 1), \quad q > 1. \end{cases}$$

(ii) For $p < q$,

$$E(m) \ll \begin{cases} 2^{-(\alpha-1/p+1/q)m} m^{(d-1)(1/q-1/\theta)_+}, & q < \infty, \\ 2^{-(\alpha-1/p)m} m^{(d-1)(1-1/\theta)_+}, & q = \infty. \end{cases}$$

Proof.

Case (i): $p \geq q$. For an arbitrary $f \in B_{p,\theta}^\alpha$, by the representation (2.30) and (2.11) we have

$$\|f - R_m(f)\|_q^\tau \ll \sum_{|k|_1 > m} \|q_k(f)\|_q^\tau$$

with any $\tau \leq \min(q, 1)$. Therefore, if $\theta \leq \min(q, 1)$, then by the inequality $\|q_k(f)\|_q \leq \|q_k(f)\|_p$ we get

$$\begin{aligned} \|f - R_m(f)\|_q &\ll \left(\sum_{|k|_1 > m} \|q_k(f)\|_q^\theta \right)^{1/\theta} \\ &\leq 2^{-\alpha m} \left(\sum_{|k|_1 > m} \{2^{\alpha|k|_1} \|q_k(f)\|_p\}^\theta \right)^{1/\theta} \\ &\ll 2^{-\alpha m} B_2(f) \ll 2^{-\alpha m}. \end{aligned}$$

Further, if $\theta > \min(q, 1)$, then

$$\|f - R_m(f)\|_q^{q^*} \ll \sum_{|k|_1 > m} \|q_k(f)\|_q^{q^*} \ll \sum_{|k|_1 > m} \{2^{\alpha|k|_1} \|q_k(f)\|_q\}^{q^*} \{2^{-\alpha|k|_1}\}^{q^*},$$

where $q^* = \min(q, 1)$. Putting $\nu = \theta/q^*$, by Hölder's inequality and the inequality $\|q_k(f)\|_q \leq \|q_k(f)\|_p$ we obtain

$$\begin{aligned} \|f - R_m(f)\|_q^{q^*} &\ll \left(\sum_{|k|_1 > m} \{2^{\alpha|k|_1} \|q_k(f)\|_q\}^{q^*\nu} \right)^{1/\nu} \left(\sum_{|k|_1 > m} \{2^{-\alpha|k|_1}\}^{q^*\nu'} \right)^{1/\nu'} \\ &\ll \{B_2(f)\}^{q^*} \{2^{-\alpha m} m^{(d-1)(1/q^*-1/\theta)}\}^{q^*} \ll \{2^{-\alpha m} m^{(d-1)(1/q^*-1/\theta)}\}^{q^*}. \end{aligned} \tag{3.3}$$

This proves the Case (i).

Case (ii): $p < q$. We first assume $q < \infty$. For an arbitrary $f \in B_{p,\theta}^\alpha$, by the representation (2.30) and Lemma 5.3 we have

$$\|f - R_m(f)\|_q^q \ll \sum_{|k|_1 > m} \{2^{(1/p-1/q)|k|_1} \|q_k(f)\|_p\}^q.$$

Therefore, if $\theta \leq q$, then

$$\begin{aligned} \|f - R_m(f)\|_q &\ll \left(\sum_{|k|_1 > m} \{2^{(1/p-1/q)|k|_1} \|q_k(f)\|_p\}^\theta \right)^{1/\theta} \\ &\ll 2^{-(\alpha-1/p+1/q)m} B_2(f) \ll 2^{-(\alpha-1/p+1/q)m}. \end{aligned}$$

Further, if $\theta > q$, then

$$\begin{aligned} \|f - R_m(f)\|_q^q &\ll \sum_{|k|_1 > m} \{2^{(1/p-1/q)|k|_1} \|q_k(f)\|_p\}^q \\ &\ll \sum_{|k|_1 > m} \{2^{\alpha|k|_1} \|q_k(f)\|_p\}^q \{2^{-(\alpha-1/p+1/q)|k|_1}\}^q. \end{aligned}$$

Hence, similarly to (3.3), we get

$$E^q(m) \ll \{2^{-(\alpha-1/p+1/q)m} m^{(d-1)(1/q-1/\theta)}\}^q.$$

When $q = \infty$, the Case (ii) can be proven analogously by use the inequality

$$\|f - R_m(f)\|_\infty \ll \sum_{|k|_1 > m} 2^{|k|_1/p} \|q_k(f)\|_p.$$

□

The following theorem for the case $\alpha = 1/p$ can be proven by use of Lemmas 2.2 and 5.3 and the inequality (2.11).

Theorem 3.2 *Let $0 < p, q < \infty$, $0 < \theta \leq \min(p, 1)$ and $\alpha = 1/p < r$. Then we have the following upper bound of $E(m)$.*

(i) For $p \geq q$,

$$E(m) \ll \begin{cases} 2^{-m/p} m^{(d-1)}, & p \geq 1, \\ 2^{-m/p} m^{(d-1)/p}, & p < 1. \end{cases}$$

(ii) For $p < q$,

$$E(m) \ll 2^{-m/q} m^{(d-1)/q}.$$

The following two theorems are a direct corollary of Lemma 3.1 and Theorems 3.1 and 3.2.

Theorem 3.3 *Let $0 < p, q, \theta \leq \infty$ and $1/p < \alpha < r$. If \bar{m} is the largest integer of m such that*

$$2^m m^{d-1} \asymp \sum_{k \in \Delta(m)} |I(k)| \leq n,$$

then we have the following upper bound of r_n and $E(\bar{m})$.

(i) *For $p \geq q$,*

$$r_n \ll E(\bar{m}) \ll \begin{cases} (n^{-1} \log^{d-1} n)^\alpha, & \theta \leq \min(q, 1), \\ (n^{-1} \log^{d-1} n)^\alpha (\log^{d-1} n)^{1/q-1/\theta}, & \theta > \min(q, 1), \quad q \leq 1, \\ (n^{-1} \log^{d-1} n)^\alpha (\log^{d-1} n)^{1-1/\theta}, & \theta > \min(q, 1), \quad q > 1. \end{cases}$$

(ii) *For $p < q$,*

$$r_n \ll E(\bar{m}) \ll \begin{cases} (n^{-1} \log^{d-1} n)^{\alpha-1/p+1/q} (\log^{d-1} n)^{(1/q-1/\theta)_+}, & q < \infty, \\ (n^{-1} \log^{d-1} n)^{\alpha-1/p} (\log^{d-1} n)^{(1-1/\theta)_+}, & q = \infty. \end{cases}$$

Theorem 3.4 *Let $0 < p, q < \infty$, $0 < \theta \leq \min(p, 1)$ and $\alpha = 1/p < r$. If \bar{m} is the largest integer of m such that*

$$2^m m^{d-1} \asymp \sum_{k \in \Delta(m)} |I(k)| \leq n,$$

then we have the following upper bound of r_n and $E(\bar{m})$.

(i) *For $p \geq q$,*

$$r_n \ll E(\bar{m}) \ll \begin{cases} (n^{-1} \log^{d-1} n)^{1/p} \log^{d-1} n, & p \geq 1, \\ (n^{-1} \log^{d-1} n)^{1/p} (\log^{d-1} n)^{1/p}, & p < 1. \end{cases}$$

(ii) *For $p < q$,*

$$r_n \ll E(\bar{m}) \ll (n^{-1} \log^{d-1} n)^{1/q} (\log^{d-1} n)^{1/q}.$$

From Theorem 3.3 and Lemma 5.1 we obtain the following theorem.

Theorem 3.5 *Let $1 \leq p, q \leq \infty$, $0 < \theta \leq \infty$ and $1/p < \alpha < r$. If \bar{m} is the largest integer of m such that*

$$2^m m^{d-1} \asymp \sum_{k \in \Delta(m)} |I(k)| \leq n,$$

then we have the following asymptotic order of r_n and $E(\bar{m})$.

(i) *For $p \geq q$ and $\theta \leq 1$,*

$$E(\bar{m}) \asymp r_n \asymp (n^{-1} \log^{d-1} n)^\alpha, \quad \begin{cases} 2 \leq q < p < \infty, \\ 1 < p = q \leq \infty. \end{cases}$$

(ii) For $1 < p < q < \infty$,

$$E(\bar{m}) \asymp r_n \asymp (n^{-1} \log^{d-1} n)^{\alpha-1/p+1/q} (\log^{d-1} n)^{(1/q-1/\theta)_+}, \begin{cases} 2 \leq p, & 2 \leq \theta \leq q, \\ q \leq 2. \end{cases}$$

4 Interpolant representations and sampling recovery

We first consider a piecewise constant interpolant representation. Let $\chi_{[0,1)}$ and $\chi_{[0,1]}$ be the characteristic functions of the half opened and closed intervals $[0, 1)$ and $[0, 1]$, respectively. For $k \in \mathbb{Z}_+$ and $s = 0, 1, \dots, 2^k - 1$, we define the system of functions $N_{k,s}$ on \mathbb{I} by

$$N_{k,s}(x) := \begin{cases} \chi_{[0,1)}(2^k x - s), & 0 \leq s < 2^k - 1, \\ \chi_{[0,1]}(2^k x - s), & s = 2^k - 1. \end{cases}$$

(In particular, $N_{0,0} = \chi_{[0,1]}$). Obviously, we have for $k > 0$ and $s = 0, 1, \dots, 2^k - 1$,

$$N_{k-1,s} = N_{k,2s} + N_{k,2s+1}.$$

We let the operator Π_k be defined for functions f on \mathbb{I} , for $k \in \mathbb{Z}_+$, by

$$\Pi_k(f) := \sum_{s=0}^{2^k-1} f(2^{-k}s) N_{k,s}, \text{ and } \Pi_{-1}(f) = 0.$$

Clearly, the linear operator Π_k is bounded in $L_\infty(\mathbb{I})$, reproduces constant functions and for any continuous function f ,

$$\|f - \Pi_k(f)\|_\infty \leq \omega_1(f, 2^{-k})_\infty,$$

and consequently, $\|f - \Pi_k(f)\|_\infty \rightarrow 0$, when $k \rightarrow \infty$. Moreover, for any $x \in \mathbb{I}$, $\Pi_k(f, x) = f(2^{-k}s)$ if x is in either the interval $[2^{-k}s, 2^{-k}(s+1))$ for $s = 0, 1, \dots, 2^k - 2$ or the interval $[2^{-k}s, 1]$ for $s = 2^k - 1$, i.e., Π_k possesses a local property. In particular, $\Pi_k(f)$ interpolates f at the points $2^{-k}s$, $s = \{0, 1, \dots, 2^k - 1\}$, that is,

$$\Pi_k(f, 2^{-k}s) = f(2^{-k}s), \quad s = 0, 1, \dots, 2^k - 1. \quad (4.1)$$

Further, we define for $k \in \mathbb{Z}_+$,

$$\pi_k(f) := \Pi_k(f) - \Pi_{k-1}(f).$$

From the definition it is easy to check that

$$\pi_k(f) = \sum_{s \in Z_1(k)} \lambda_{k,s}^1(f) \varphi_{k,s}^1,$$

where $Z_1(0) := \{0\}$, $Z_1(k) := \{0, 1, \dots, 2^{k-1} - 1\}$ for $k > 0$,

$$\varphi_{k,s}^1(x) := N_{k,2s+1}(x), \quad k > 0, \text{ and } \varphi_{0,0}^1(x) := N_{0,0}(x),$$

and

$$\lambda_{k,s}^1(f) := \Delta_{2^{-k}}^1(f, 2^{-k+1}s), \quad k > 0, \quad \text{and} \quad \lambda_{0,0}^1(f) := f(0).$$

We now can see that every $f \in C(\mathbb{I})$ is represented by the series

$$f = \sum_{k \in \mathbb{Z}_+} \sum_{s \in Z_1(k)} \lambda_{k,s}^1(f) \varphi_{k,s}^1, \quad (4.2)$$

converging in the norm of $L_\infty(\mathbb{I})$.

Next, let us revisit the univariate piecewise linear (nodal) quasi-interpolant for functions on \mathbb{R} defined in (2.4) with $M(x) = (1 - |x|)_+$ ($r = 2$). Consider the generated from it by the formula (2.5) quasi-interpolant for functions on \mathbb{I}

$$Q_k(f, x) = \sum_{s \in J(k)} f(2^{-k}s) M_{k,s}(x), \quad (4.3)$$

and the related quasi-interpolant representation

$$f = \sum_{k \in \mathbb{Z}_+} q_k(f) = \sum_{k \in \mathbb{Z}_+} \sum_{s \in J(k)} c_{k,s}(f) M_{k,s}, \quad (4.4)$$

where we recall that $J(k) := \{s \in \mathbb{Z} : 0 \leq s \leq 2^k\}$ is the set of s for which $M_{k,s}$ do not vanish identically on \mathbb{I} . From the equality $M_{k,s}(2^{-k}s') = \delta_{s,s'}$ one can see that $Q_k(f)$ interpolates f at the dyadic points $2^{-k}s$, $s \in J(k)$, i.e.

$$Q_k(f, 2^{-k}s) = f(2^{-k}s), \quad s \in J(k). \quad (4.5)$$

Because of the interpolation property (4.1) and (4.5), the operators Π_k and Q_k are interpolants. Therefore, the representations (4.2) and (4.4) are interpolant representations. We will see that the interpolant representation (4.4) coincides with the classical Faber-Schauder series. The univariate Faber-Schauder system of functions is defined by

$$\mathcal{F} := \{\varphi_{k,s}^2 : s \in Z_2(k), \quad k \in \mathbb{Z}_+\},$$

where $Z_2(0) := \{0, 1\}$ and $Z_2(k) := \{0, 1, \dots, 2^{k-1} - 1\}$ for $k > 0$,

$$\varphi_{0,0}^2(x) := M_{0,0}(x), \quad \varphi_{0,1}^2(x) := M_{0,1}(x), \quad x \in \mathbb{I},$$

(an alternative choice is $\varphi_{0,1}(x) := 1$), and for $k > 0$ and $s \in Z(k)$

$$\varphi_{k,s}^2(x) := M_{k,2s+1}(x), \quad x \in \mathbb{I}.$$

It is known that \mathcal{F} is a basis in $C(\mathbb{I})$. (See [15] for details about the Faber-Schauder system.)

By a direct computation we have for the component functions $q_k(f)$ in the piecewise linear quasi-interpolant representation (4.4):

$$q_k(f) = \sum_{s \in Z_2(k)} \lambda_{k,s}^2(f) \varphi_{k,s}^2(x). \quad (4.6)$$

where

$$\lambda_{k,s}^2(f) := -\frac{1}{2}\Delta_{2^{-k}}^2 f(2^{-k+1}s), \quad k > 0, \quad \text{and} \quad \lambda_{0,s}^2(f) := f(s).$$

Hence, the interpolant representation (4.4) can be rewritten as the Faber-Schauder series:

$$f = \sum_{k \in \mathbb{Z}_+} q_k(f) = \sum_{k \in \mathbb{Z}_+} \sum_{s \in Z_2(k)} \lambda_{k,s}^2(f) \varphi_{k,s}^2,$$

and for any continuous function f on \mathbb{I} ,

$$\|f - Q_k(f)\|_\infty \leq \omega_2(f, 2^{-k})_\infty.$$

Put $Z_r^d(k) := \prod_{i=1}^d Z_r(k_i)$, $r = 1, 2$. For $k \in \mathbb{Z}_+^d$, $s \in Z_r^d(k)$, define

$$\varphi_{k,s}^r(x) := \prod_{i=1}^d \varphi_{k_i, s_i}^r(x_i),$$

and $\lambda_{k,s}^r(f)$ in the manner of the definition (2.7) by

$$\lambda_{k,s}^r(f) := \lambda_{k_1, s_1}^r((\lambda_{k_2, s_2}^r(\dots \lambda_{k_d, s_d}^r(f))).$$

Theorem 4.1 *Let $r = 1, 2$, $0 < p, \theta \leq \infty$ and $1/p < \alpha < r$. Then there hold the following assertions.*

(i) *A function $f \in MB_{p,\theta}^\alpha$ can be represented by the series*

$$f = \sum_{k \in \mathbb{Z}_+^d} \sum_{s \in Z_r^d(k)} \lambda_{k,s}^r(f) \varphi_{k,s}^r, \quad (4.7)$$

converging in the quasi-norm of $MB_{p,\theta}^\alpha$. Moreover, we have

$$B^*(f) := \left(\sum_{k \in \mathbb{Z}_+^d} \left\{ 2^{(\alpha-1/p)|k|_1} \left(\sum_{s \in Z_r^d(k)} |\lambda_{k,s}^r(f)|^p \right)^{1/p} \right\}^\theta \right)^{1/\theta} \leq CB(f).$$

(ii) *If in addition, $r = 2$ and $\alpha < \min(2, 1 + 1/p)$, then a continuous function f on \mathbb{I}^d belongs to the Besov space $MB_{p,\theta}^\alpha$ if and only if f can be represented by the series (4.7). Moreover, the Besov quasi-norm $B(f)$ is equivalent to the discrete quasi-norm $B^*(f)$.*

Proof. If $r = 2$, from the definition and (4.6) we can derive that for functions on \mathbb{I}^d and $k \in \mathbb{Z}_+^d$, the component function $q_k(f)$ in the interpolant representation (2.30) related to the interpolant (4.3), can be rewritten as

$$q_k(f) = \sum_{s \in Z_2^d(k)} \lambda_{k,s}^2(f) \varphi_{k,s}^2(x). \quad (4.8)$$

Therefore, Theorem 4.1 is the rewritten Theorem 2.1. This does not hold for the case $r = 1$. However, the last case can be proven in a way completely similar to the proof of Theorem 2.1 by using the above mentioned properties of the functions $\varphi_{k,s}^1$ and operators Π_k . \square

For $m \in \mathbb{Z}_+$, we have by (4.8)

$$R_m^r(f) = R_m(f) = \sum_{k \in \Delta(m)} \sum_{s \in Z_r^d(k)} \lambda_{k,s}^r(f) \varphi_{k,s}^r.$$

For functions f on \mathbb{I}^d , R_m^r defines a linear sampling algorithm of the form (1.1) on the grid $G_r^d(m)$ where $G_r^d(m) := \{2^{-k}s : k \in \Delta(m), s \in I_r^d(k)\}$, $I_1^d(k) := \{s \in \mathbb{Z}_+^d : 0 \leq s_i \leq 2^{k_i} - 1, i \in N[d]\}$, $I_2^d(k) := I^d(k)$. More precisely,

$$R_m^r(f) = L_n^r(f) = \sum_{k \in \Delta(m)} \sum_{j \in I_r^d(k)} f(2^{-k}j) \psi_{k,j}^r,$$

where

$$n := \sum_{k \in \Delta(m)} |I_r^d(k)| \asymp 2^m m^{d-1};$$

$$\psi_{k,s}^r(x) = \prod_{i=1}^d \psi_{k_i, s_i}^r(x_i), \quad k \in \mathbb{Z}_+^d, \quad s \in I_r^d(k),$$

and the univariate functions $\psi_{k,s}^r$ are defined by

$$\psi_{k,s}^1 = \begin{cases} \varphi_{k,s}^1, & k = 0, \quad s = 0, \\ \varphi_{k,j}^1, & k > 0, \quad s = 2j + 1, \\ -\varphi_{k,j}^1, & k > 0, \quad s = 2j, \end{cases}$$

and

$$\psi_{k,s}^2 = \begin{cases} \varphi_{k,s}^2, & k = 0, \\ -\frac{1}{2}\varphi_{k,0}^2, & k > 0, \quad s = 0, \\ \varphi_{k,j}^2, & k > 0, \quad s = 2j + 1, \\ -\frac{1}{2}(\varphi_{k,j}^2 + \varphi_{k,j-1}^2), & k > 0, \quad s = 2j, \\ -\frac{1}{2}\varphi_{k,2^{k-1}-1}^2, & k > 0, \quad s = 2^k. \end{cases}$$

From the interpolation properties (4.1) and (4.5), the equality $\varphi_{k,s}^r(2^{-k}s') = \delta_{s,s'}$ one can easily verify that $R_m^r(f)$ interpolates f at the grid $G_r^d(m)$, i.e.,

$$R_m^r(f, x) = f(x), \quad x \in G_r^d(m).$$

Theorem 4.2 *Let $r = 2$, $0 < p, q, \theta \leq \infty$, and $1/p < \alpha < \min(2, 1 + 1/p)$. Then we have*

(i) *For $p \geq q$,*

$$E(m) \gg 2^{-\alpha m} m^{(d-1)(1-1/\theta)_+}.$$

(ii) For $p < q$,

$$E(m) \gg 2^{-(\alpha-1/p+1/q)m} m^{(d-1)(1/q-1/\theta)_+}.$$

Proof. Put $\Gamma(m) := \{k \in \mathbb{N}^d : |k|_1 = m+1\}$. Let the half-open d -cube $I(k, s)$ be defined by $I(k, s) := \prod_{i=1}^d [s_i 2^{-(k_i-1)}, (s_i+1) 2^{-(k_i-1)})$. Notice that $I(k, s) \subset \mathbb{I}^d$ and $I(k, s) \cap I(k, s') = \emptyset$ for $s \neq s'$. Moreover, if $0 < \nu \leq \infty$, for $k \in \Gamma(m)$, $s \in Z^d(k)$,

$$\|\varphi_{k,s}^2\|_\nu = \left(\int_{I(k,s)} |\varphi_{k,s}^2(x)|^\nu dx \right)^{1/\nu} \asymp 2^{-m/\nu}, \quad (4.9)$$

with the change to sup when $\nu = \infty$, and

$$\left\| \sum_{s \in Z_2^d(k)} \varphi_{k,s}^2 \right\|_\nu \asymp 1. \quad (4.10)$$

Case (i): For an integer $m \geq 1$, we take the functions

$$g_1 := C_1 2^{-\alpha m} \sum_{s \in Z_2^d(\bar{k})} \varphi_{k,s}^2 \quad (4.11)$$

with some $\bar{k} \in \Gamma(m)$, and

$$g_2 := C_2 2^{-\alpha m} m^{-(d-1)/\theta} \sum_{k \in \Gamma(m)} \sum_{s \in Z^d(k)} \varphi_{k,s}^2. \quad (4.12)$$

Notice that the right side of (4.11) and (4.12) defines the series (4.7) of g_i , $i = 1, 2$. By Theorem 4.1 and (4.10) we can choose constants C_i so that $g_i \in B_{p,\theta}^\alpha$ for all $m \geq 1$ and $i = 1, 2$. It is easy to verify that $g_i - R_m^2(g_i) = g_i$ $i = 1, 2$. We have by (4.10)

$$E(m) \geq \|g_1\|_q \gg 2^{-\alpha m}$$

if $\theta \leq 1$, and

$$E(m) \geq \|g_2\|_q \geq \|g_2\|_{q^*} \gg 2^{-\alpha m} m^{(d-1)(1-1/\theta)}$$

if $\theta > 1$, where $q^* := \min(q, 1)$.

Case (ii): Let $s(k) \in \mathbb{Z}_+^d$ be defined by $s(k)_i = \sum_{j=1}^{k_i-2} 2^j$ if $k_i > 2$, and $s(k)_i = 0$ if $k_i = 2$ for $i = 1, \dots, d$, and $\Gamma^*(m) := \{k \in \Gamma(m) : k_i \geq 2, i = 1, \dots, d\}$. For an integer $m \geq 2$, we take the functions

$$g_3 = C_3 2^{-(\alpha-1/p)m} \varphi_{k^*, s(k^*)}^2 \quad (4.13)$$

with some $k^* \in \Gamma^*(m)$, and

$$g_4 = C_4 2^{-(\alpha-1/p)m} m^{-(d-1)/\theta} \sum_{k \in \Gamma^*(m)} \varphi_{k, s(k)}^2. \quad (4.14)$$

Similarly to the functions g_i , $i = 1, 2$, the right side of (4.13) and (4.14) defines the series (4.7) of g_i , $i = 3, 4$, and we can choose constants C_i so that $g_i \in B_{p,\theta}^\alpha$ for all $m \geq 2$ and $i = 3, 4$. Obviously, $g_i - R_m^2(g_i) = g_i$, $i = 3, 4$. We have by (4.9)

$$E(m) \geq \|g_3\|_q \gg 2^{-(\alpha-1/p+1/q)m}$$

if $\theta \geq q$, and

$$E(m) \geq \|g_4\|_q \gg 2^{-(\alpha-1/p+1/q)m} m^{(d-1)(1/q-1/\theta)}$$

if $\theta < q$. \square

From Theorems 3.1 and 4.2 we obtain

Theorem 4.3 *Let $r = 2$, $0 < p, q, \theta \leq \infty$, and $1/p < \alpha < \min(2, 1 + 1/p)$. Then we have*

(i) *For $p \geq q$,*

$$E(m) \asymp \begin{cases} 2^{-\alpha m}, & \theta \leq \min(q, 1), \\ 2^{-\alpha m} m^{(d-1)(1-1/\theta)}, & \theta > 1, q \geq 1. \end{cases}$$

(ii) *For $p < q < \infty$,*

$$E(m) \asymp 2^{-(\alpha-1/p+1/q)m} m^{(d-1)(1/q-1/\theta)_+}.$$

Notice that Theorem 4.3(i) has been proven in [22] for the $1 \leq p = q = \theta \leq \infty$.

5 Appendix

Lemma 5.1 *Let $1 \leq p, q \leq \infty$, $0 < \theta \leq \infty$ and $\alpha > (1/p - 1/q)_+$. Then we have the following asymptotic order of $\lambda_n(B_{p,\theta}^\alpha)_q$.*

(i) *For $p \geq q$,*

$$\lambda_n(B_{p,\theta}^\alpha)_q \asymp \begin{cases} (n^{-1} \log^{d-1} n)^\alpha, & \theta \leq 2 \leq q \leq p < \infty, \\ (n^{-1} \log^{d-1} n)^\alpha, & \theta \leq 1, p = q = \infty, \\ (n^{-1} \log^{d-1} n)^\alpha, & 1 < p = q \leq 2, \theta \leq q, \\ (n^{-1} \log^{d-1} n)^\alpha (\log^{d-1} n)^{1/q-1/\theta}, & 1 < p = q \leq 2, \theta > q \\ (n^{-1} \log^{d-1} n)^\alpha (\log^{d-1} n)^{1/2-1/\theta}, & \theta > 2. \end{cases}$$

(ii) *For $1 < p < q < \infty$,*

$$\lambda_n(B_{p,\theta}^\alpha)_q \asymp \begin{cases} (n^{-1} \log^{d-1} n)^{\alpha-1/p+1/q}, & 2 \leq p, 2 \leq \theta \leq q, \\ (n^{-1} \log^{d-1} n)^{\alpha-1/p+1/q} (\log^{d-1} n)^{(1/q-1/\theta)_+}, & q \leq 2. \end{cases}$$

Proof. This lemma was proven in [14], [20] except the cases $\theta \leq 2 \leq q \leq p < \infty$ and $\theta \leq 1$, $p = q = \infty$ which can be obtained from the asymptotic order [20]

$$\lambda_n(B_{p,\theta}^\alpha)_q \asymp (n^{-1} \log^{d-1} n)^\alpha, \begin{cases} 1 \leq \theta \leq 2 \leq q \leq p < \infty, \\ \theta = 1, p = q = \infty, \end{cases}$$

and the equalities $\lambda_n(W)_q = \lambda_n(\text{co}W)_q$ and $\text{co}B_{p,\theta}^\alpha = B_{p,\max(\theta,1)}^\alpha$, where $\text{co}W$ denotes the convex hull of W . \square

For $\mathbf{p} = (p_1, \dots, p_d) \in (0, \infty)^d$, we defined the mixed integral quasi-norm $\|\cdot\|_{\mathbf{p}}$ for functions on \mathbb{I}^d as follows

$$\|f\|_{\mathbf{p}} := \left(\int_{\mathbb{I}} \left(\cdots \int_{\mathbb{I}} \left(\int_{\mathbb{I}} |f(x)|^{p_1} dx_1 \right)^{p_2/p_1} dx_2 \cdots \right)^{p_d/p_{d-1}} dx_d \right)^{1/p_d},$$

and put $1/\mathbf{p} := (1/p_1, \dots, 1/p_d)$. If $\mathbf{p}, \mathbf{q} \in (0, \infty)^d$ and $\mathbf{p} \leq \mathbf{q}$, then there holds Nikol'skii's inequality for any $f \in \Sigma_r^d(k)$,

$$\|f\|_{\mathbf{q}} \leq C 2^{|(1/\mathbf{p}-1/\mathbf{q})k|_1} \|f\|_{\mathbf{p}} \quad (5.1)$$

with constant C depending on $\mathbf{p}, \mathbf{q}, d$ only. This inequality can be proven by a generalization of the Jensen's inequality for mixed norms and the following equivalences of the mixed integral quasi-norm $\|\cdot\|_{\mathbf{p}}$. For all $k \in \mathbb{Z}_+^d$ and all $f \in \Sigma_r^d(k)$ of the form (2.14),

$$\|f\|_{\mathbf{p}} \asymp \prod_{i=1}^d 2^{-k_i/p_i} \|\{a_s\}\|_{\mathbf{p},k},$$

where

$$\|\{a_s\}\|_{\mathbf{p},k} := \left(\sum_{s_d \in J(k_d)} \left(\cdots \sum_{s_2 \in J(k_2)} \left(\sum_{s_1 \in J(k_1)} |a_s|^{p_1} \right)^{p_2/p_1} \cdots \right)^{p_d/p_{d-1}} \right)^{1/p_d}.$$

Lemma 5.2 *Let $0 < p < q < \infty$, $\delta = 1/2 - p/(p+q)$. If $k, s \in \mathbb{Z}_+^d$, then for any $\varphi_k \in \Sigma_r^d(k)$ and $\varphi_s \in \Sigma_r^d(s)$, there holds the inequality*

$$\int_{\mathbb{I}^d} |\varphi_k(x) \varphi_s(x)|^{q/2} dx \leq C A_k A_s 2^{-\delta|k-s|_1},$$

with some constant C depending at most on p, q, d , where

$$A_k := \left(2^{(1/p-1/q)|k|_1} \|\varphi_k\|_p \right)^{q/2}.$$

Proof. Put $\nu := (p+q)/p$. Then $\delta = 1/2 - 1/\nu$ and $2 < \nu < \infty$. Let ν' be given by $1/\nu + 1/\nu' = 1$. Then $1 < \nu' < 2$. Let $\mathbf{u}, \mathbf{v} \in (0, \infty)^d$ be defined by $\mathbf{u} := q\mathbf{v}/2$ and $v_i = \nu$ if $k_i \geq s_i$ and $v_i = \nu'$ if $k_i < s_i$ for $i = 1, \dots, d$. Let \mathbf{u}' and \mathbf{v}' be given by $1/\mathbf{u} + 1/\mathbf{u}' = \mathbf{1}$ and $1/\mathbf{v} + 1/\mathbf{v}' = \mathbf{1}$, respectively.

Notice that $\mathbf{v} \in (1, \infty)^d$. Applying Hölder's inequality for the mixed norm $\|\cdot\|_{\mathbf{v}}$ to the functions $|\varphi_k|^{q/2}$ and $|\varphi_s|^{q/2}$, we obtain

$$\int_{\mathbb{I}^d} |\varphi_k(x) \varphi_s(x)|^{q/2} dx \leq \| |\varphi_k|^{q/2} \|_{\mathbf{v}} \| |\varphi_s|^{q/2} \|_{\mathbf{v}'} = \|\varphi_k\|_{\mathbf{u}}^{q/2} \|\varphi_s\|_{\mathbf{u}'}^{q/2}. \quad (5.2)$$

Since $\mathbf{u} > p\mathbf{1}$ and $\mathbf{u}' > p\mathbf{1}$, by the inequality (5.1) we have

$$\|\varphi_k\|_{\mathbf{u}} \leq 2^{[(1/p-1/\mathbf{u})k]_1} \|\varphi_k\|_p, \quad \|\varphi_s\|_{\mathbf{u}'} \leq 2^{[(1/p-1/\mathbf{u}')s]_1} \|\varphi_s\|_p. \quad (5.3)$$

From (5.2) and (5.3) we prove the lemma. \square

Lemma 5.3 *Let $0 < p < q < \infty$ and $g \in L_q$ be represented by the series*

$$g = \sum_{k \in \mathbb{Z}_+^d} g_k, \quad g_k \in \Sigma_r^d(k).$$

Then there holds the inequality

$$\|g\|_q \leq C \left(\sum_{k \in \mathbb{Z}_+^d} \|2^{(1/p-1/q)|k|_1} g_k\|_p^q \right)^{1/q}, \quad (5.4)$$

with some constant C depending at most on p, d , whenever the right side is finite.

Proof. It is enough to prove the inequality (5.4) for g of the form

$$g = \sum_{k \leq m} g_k, \quad g_k \in \Sigma_r^d(k),$$

for any $m \in \mathbb{Z}_+^d$.

Put $n := [q] + 1$. Then $0 < q/n \leq 1$. By Jensen's inequality we have

$$\begin{aligned} \left| \sum_{k \leq m} g_k(x) \right|^q &= \left(\left| \sum_{k \leq m} g_k(x) \right|^{q/n} \right)^n \\ &\leq \left(\sum_{k \leq m} |g_k(x)|^{q/n} \right)^n = \sum_{k^1 \leq m} \cdots \sum_{k^n \leq m} \prod_{j=1}^n |g_{k^j}(x)|^{q/n}. \end{aligned}$$

consequently,

$$\|g\|_q^q \leq \sum_{k^1 \leq m} \cdots \sum_{k^n \leq m} \int_{\mathbb{I}^d} \prod_{j=1}^n |g_{k^j}(x)|^{q/n} dx. \quad (5.5)$$

By use of the identity

$$\prod_{j=1}^n a_j = \left(\prod_{i \neq j} a_i a_j \right)^{1/2(n-1)} \quad (5.6)$$

for non-negative numbers a_1, \dots, a_n , we get

$$J := \int_{\mathbb{I}^d} \prod_{j=1}^n |g_{kj}(x)|^{q/n} dx = \int_{\mathbb{I}^d} \prod_{i \neq j} |g_{ki}(x) g_{kj}(x)|^{q/2n(n-1)} dx.$$

Hence, applying Hölder's inequality to $n(n-1)$ functions in the right side of the last equality, Lemma 5.2 and (5.6) gives

$$\begin{aligned} J &\leq \prod_{i \neq j} \left(\int_{\mathbb{I}^d} |g_{ki}(x) g_{kj}(x)|^{q/2} dx \right)^{1/n(n-1)} \leq \prod_{i \neq j} \left(A_{ki} A_{kj} 2^{-\delta|k^i - k^j|_1} \right)^{1/n(n-1)} \\ &= \prod_{i \neq j} (A_{ki} A_{kj})^{1/n(n-1)} \left\{ \left(\prod_{i \neq j} \prod_{i'=1}^n 2^{-\delta|k^i - k^{i'}|_1} \prod_{j'=1}^n 2^{-\delta|k^j - k^{j'}|_1} \right)^{1/2(n-1)} \right\}^{1/n(n-1)} \\ &= \left\{ \prod_{i \neq j} (A_{ki} A_{kj})^{2/n} \left(\prod_{i'=1}^n 2^{-\delta|k^i - k^{i'}|_1} \prod_{j'=1}^n 2^{-\delta|k^j - k^{j'}|_1} \right)^{1/n(n-1)} \right\}^{1/2(n-1)} \\ &= \prod_{j=1}^n A_{kj}^{2/n} \left(\prod_{i=1}^n 2^{-\delta|k^j - k^i|_1} \right)^{1/n(n-1)} = \left(\prod_{j=1}^n A_{kj}^2 \prod_{i=1}^n 2^{-\lambda\delta|k^j - k^i|_1} \right)^{1/n}, \end{aligned}$$

where $\lambda := \delta/(n-1) > 0$. Therefore, from (5.5) and Hölder's inequality we obtain

$$\begin{aligned} \|g\|_q^q &\leq \sum_{k^1 \leq m} \cdots \sum_{k^n \leq m} \left(\prod_{j=1}^n A_{kj}^2 \prod_{i=1}^n 2^{-\lambda\delta|k^j - k^i|_1} \right)^{1/n} \\ &\leq \prod_{j=1}^n \left(\sum_{k^1 \leq m} \cdots \sum_{k^n \leq m} A_{kj}^2 \prod_{i=1}^n 2^{-\lambda\delta|k^j - k^i|_1} \right)^{1/n} =: \prod_{j=1}^n B_j. \end{aligned} \tag{5.7}$$

We have

$$\begin{aligned} B_j &= \sum_{k^j \leq m} A_{kj}^2 \sum_{k^1 \leq m} \cdots \sum_{k^{j-1} \leq m} \sum_{k^{j+1} \leq m} \cdots \sum_{k^n \leq m} \prod_{i=1}^n 2^{-\lambda\delta|k^j - k^i|_1} \\ &= \sum_{k^j \leq m} A_{kj}^2 \left(\sum_{s \leq m} 2^{-\lambda\delta|k^j - s|_1} \right)^{n-1} \leq C \sum_{k^j \leq m} A_{kj}^2. \end{aligned}$$

Using this estimate for B_j , we can continue (5.7) and finish the estimation of $\|g\|_q^q$ as follows.

$$\begin{aligned} \|g\|_q^q &\leq \prod_{j=1}^n B_j^{1/n} \leq C \sum_{k \leq m} A_k^2 \\ &= C \sum_{k \leq m} \|2^{(1/p-1/q)|k|_1} g_k\|_p^q. \end{aligned}$$

Thus, the proof of the lemma is completed. \square

Remark A trigonometric polynomial version of Lemma 5.3 was proven in [25] for $1 \leq p < q < \infty$.

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